Virial theorem and dynamical evolution of self-gravitating Brownian particles in an unbounded domain. II. Inertial models

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We propose a general kinetic and hydrodynamic description of self-gravitating Brownian particles in d dimensions. We go beyond the usual approximations by considering inertial effects and finite-N effects while previous works use a mean-field approximation valid in a proper thermodynamic limit ($N\to+\infty$) and consider an overdamped regime ($\xi\to+\infty$). We recover known models in some particular cases of our general description. We derive the expression of the virial theorem for self-gravitating Brownian particles and study the linear dynamical stability of isolated clusters of particles and uniform systems by using techniques introduced in astrophysics. We investigate the influence of the equation of state, of the dimension of space, and of the friction coefficient on the dynamical stability of the system. We obtain the exact expression of the critical temperature T_c for a multicomponents self-gravitating Brownian gas in $d=2$. We also consider the limit of weak frictions, $\xi\to 0$, and derive the orbit-averaged Kramers equation.

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I. INTRODUCTION

Self-gravitating systems such as globular clusters and galaxies can be considered as a collection of N stars in gravitational interaction whose dynamics is described by the Hamilton equations of motion [1]. In statistical mechanics, this situation is associated with the microcanonical ensemble where the energy and particle number are fixed [2,3]. In a recent series of papers [4–11], we have proposed to consider a system of self-gravitating Brownian particles which are subject, in addition to the gravitational force, to a friction and a noise. Their dynamics is described by N-coupled stochastic Langevin equations. In statistical mechanics, this situation is associated with the canonical ensemble where the temperature and particle number are fixed [12]. In previous papers, we have considered a mean-field approximation valid in a proper thermodynamic limit with $N\to+\infty$ and, in order to simplify the problem, we have studied a limit of strong friction $\xi\to+\infty$ or, equivalently, a large-time regime $t\gg\xi^{-1}$. In these approximations, the problem is reduced to the study of the Smoluchowski-Poisson (SP) system. We have also introduced a generalized class of stochastic processes and kinetic equations in which the diffusion coefficient (or the friction and mobility) is allowed to depend on the concentration of particles [13–17]. This can model microscopic constraints (e.g., close packing) acting on the particles when their local concentration becomes large. The evolution of the system is then described by the generalized Smoluchowski-Poisson (GSP) system involving a barotropic equation of state $p(\rho)$ specified by the stochastic model. This mean-field nonlinear Fokker-Planck (MFNFP) equation admits a Lyapunov functional, determined by the equation of state, which can be interpreted as a generalized free energy [13]. Thus, this model is associated with a notion of (effective) "generalized thermodynamics" in μ space. In the classical case where the diffusion coefficient is constant, we recover an isothermal equation of state $p=\rho k_B T/m$ associated with the Boltzmann free energy but more general equations of state can be considered. Interestingly, the same type of drift-diffusion equations are encountered in mathematical biology to describe the chemotactic aggregration of bacterial populations, in relation with the Keller-Segel model [18–20]. The analogy between self-gravitating Brownian particles and bacterial populations is developed in [9].

Here, we propose to generalize these models so as to take into account finite-N effects and inertial effects (finite friction ξ). We thus propose a general kinetic and hydrodynamic description of self-gravitating Brownian particles starting directly from a system of N coupled Langevin equations of motion with long-range attractive interactions. We shall extend the techniques developed in astrophysics to our problem of self-gravitating Brownian particles. In particular, we shall derive the appropriate expression of the virial theorem for these systems and study their linear dynamical stability by a method similar to that developed by Eddington [21] and Ledoux and Walraven [22] for barotropic stars described by the Euler equations. We shall make the link between parabolic and hyperbolic models by considering an intermediate model, taking into account the inertia of the particles as well as a friction force. The Euler equations are recovered for $\xi=0$ and the Smoluchowski equation is obtained in the limit $\xi\to+\infty$.

The paper is organized as follows. In Sec. II, we introduce general kinetic and hydrodynamic models of self-gravitating Brownian particles starting from coupled Langevin equations. We derive the N-body Fokker-Planck equation (Sec. II A), the mean-field Kramers equation (Sec. II B), and the generalized mean-field Kramers equation (Sec. II C). Then, we take the hydrodynamic moments of these equations and derive the damped Jeans equations (Sec. II D) and the

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damped barotropic Euler equations (Sec. II E) by closing the hierarchy of moments with a local thermodynamical equilibrium (LTE) hypothesis. We obtain the mean-field Smoluchowski equation in a strong-friction limit $\xi \to +\infty$ and, in Sec. II F, we derive the orbit-averaged Kramers equation as a weak-friction limit $\xi \to 0$. In Sec. III, we establish the general expression of the virial theorem for self-gravitating Brownian particles from the damped Jeans equations (Sec. III A) and from the damped Euler equations (Sec. III B). We also consider the effect of correlations due to finite-N effects (Sec. III C). In Sec. IV, we study the linear dynamical stability of an inhomogeneous stationary solution of the damped barotropic Euler-Poisson system and investigate the effect of the friction coefficient on the evolution of the perturbation. In Appendix A we give a short complement concerning the stability of polytropic systems, and in Appendix B we derive the exact expression of the virial theorem starting directly from the stochastic equations of motion. We show that the virial theorem takes a very simple form in dimensions $d=2$ and $d=4$ and analyze the consequences of this simplification. In Appendix C, we study the linear dynamical stability of homogeneous stationary solutions of the damped barotropic Euler equations (for an arbitrary potential of interaction) and obtain a generalization of the Jeans instability criterion. In the Conclusion, we discuss the different regimes in the evolution of Hamiltonian and Brownian systems with long-range interactions [12], distinguishing the phase of violent collisionless relaxation, the collisional evolution due to finite-N effects, and the "collisional" evolution due to *imposed* friction and stochastic forces for Brownian systems.

II. KINETIC AND HYDRODYNAMIC MODELS OF SELF-GRAVITATING BROWNIAN PARTICLES

The Smoluchowski-Poisson system that has been extensively studied in previous papers [4–11] describes a gas of self-gravitating Brownian particles in a mean-field approximation (valid for $N \to +\infty$) and in a strong-friction limit $\xi \to +\infty$. In this section, we introduce more general kinetic and hydrodynamic models of self-gravitating Brownian particles that go beyond these approximations.

A. N-body Fokker-Planck equation

Basically, a system of self-gravitating Brownian particles is described by the N coupled stochastic equations

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \tag{1}$$

$$\frac{d\mathbf{v}_i}{dt} = -\xi \mathbf{v}_i - m\nabla_i U(\mathbf{r}_1, \ldots, \mathbf{r}_N) + \sqrt{2D}\mathbf{R}_i(t), \tag{2}$$

where $-\xi \mathbf{v}_i$ is a friction force and $\mathbf{R}_i(t)$ is a Gaussian white noise satisfying $\langle \mathbf{R}_i(t) \rangle = \mathbf{0}$ and $\langle R_{a,i}(t) R_{b,j}(t') \rangle = \delta_{ij}\delta_{ab}\delta(t-t')$, where $a, b = 1, \ldots, d$ refer to the coordinates of space and $i, j = 1, \ldots, N$ to the particles. We define the inverse temperature $\beta = 1/k_B T$ through the Einstein relation $\xi = D\beta m$ (see below). For $\xi = 0$ and $D = 0$, Eqs. (1) and (2) reduce to the Newton-Hamilton equations of motion describing the ordinary self-gravitating gas with a Hamiltonian

$$H = \sum_{i=1}^{N} \frac{1}{2} m v_i^2 + m^2 U(\mathbf{r}_1, \ldots, \mathbf{r}_N), \tag{3}$$

where $U(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \Sigma_{i<j} u(\mathbf{r}_i - \mathbf{r}_j)$ and $u(\mathbf{r}_i - \mathbf{r}_j) = -G/[(d-2)|\mathbf{r}_i - \mathbf{r}_j|^{(d-2)}]$ denotes the gravitational potential of interaction in d dimensions [$u(\mathbf{r}_i - \mathbf{r}_j) = G \ln|\mathbf{r}_i - \mathbf{r}_j|$ for $d=2$]. In this paper, we shall be particularly interested in the gravitational interaction, but we stress that our formalism remains valid for a more general class of binary potentials of interaction of the form $u(|\mathbf{r}_i - \mathbf{r}_j|)$. The evolution of the N-body distribution function is governed by the Fokker-Planck equation [12]

$$\frac{\partial P_N}{\partial t} + \sum_{i=1}^{N} \left(\mathbf{v}_i \cdot \frac{\partial P_N}{\partial \mathbf{r}_i} + \mathbf{F}_i \cdot \frac{\partial P_N}{\partial \mathbf{v}_i} \right) = \sum_{i=1}^{N} \frac{\partial}{\partial \mathbf{v}_i} \cdot \left[D \frac{\partial P_N}{\partial \mathbf{v}_i} + \xi P_N \mathbf{v}_i \right]. \tag{4}$$

In the absence of friction and diffusion ($\xi = D = 0$), it reduces to the Liouville equation. The Liouville equation conserves the energy $\langle E \rangle = \int P_N H \Pi_i d\mathbf{r}_i d\mathbf{v}_i$ and the Gibbs entropy $S = -k_B \int P_N \ln P_N d\mathbf{r}_1 d\mathbf{v}_1 \cdots d\mathbf{r}_N d\mathbf{v}_N$ (more generally, any functional of P_N) defined on Γ space. This corresponds to a microcanonical description. Alternatively, in the Brownian model, the temperature T is fixed (instead of the energy) and the Fokker-Planck equation (4) decreases the Gibbs free energy $F = \langle E \rangle - TS$ which can be written, explicitly,

$$F[P_N] = \int P_N H \prod_i d\mathbf{r}_i d\mathbf{v}_i + k_B T \int P_N \ln P_N \prod_i d\mathbf{r}_i d\mathbf{v}_i. \tag{5}$$

This corresponds to a canonical description. One has

$$\dot{F} = -\sum_{i=1}^{N} \int \frac{1}{\xi P_N} \left(D \frac{\partial P_N}{\partial \mathbf{v}_i} + \xi P_N \mathbf{v}_i \right)^2 d\mathbf{r}_1 d\mathbf{v}_1 \cdots d\mathbf{r}_N d\mathbf{v}_N \leqslant 0. \tag{6}$$

Therefore, the free energy plays the role of a Lyapunov functional for the N-body Fokker-Planck equation. At equilibrium, $\dot{F} = 0$ implying that the right-hand side (RHS) of Eq. (4) vanishes. The LHS (advective term) must also vanish independently. From these two requirements we find that the stationary solution of Eq. (4) is the canonical distribution[1]

[1] In order to properly define a *strict* statistical equilibrium state for self-gravitating systems, one has to introduce a small-scale regularization; otherwise, $F[P_N]$ has no minimum and Eq. (7) is not normalizable. Thus, in Eq. (7), it is implicitly understood that U is a regularized potential. Note that *physical* statistical equilibrium states unaffected by the small-scale regularization exist in the form of long-lived metastable structures that are *local* minima of the Boltzmann mean-field free energy $F_B[f]$ defined in Eq. (13). We refer to [23] for a physical discussion of these issues.

$$P_N(\mathbf{r}_1,\mathbf{v}_1,\ldots,\mathbf{r}_N,\mathbf{v}_N)=\frac{1}{Z(\beta)}e^{-\beta m[\Sigma_{i=1}^N v_i^2/2+mU(\mathbf{r}_1,\ldots,\mathbf{r}_N)]}, \quad (7)$$

provided that the coefficients of diffusion and friction are connected by the Einstein relation $\xi=D\beta m$. The partition function $Z(\beta)$ is determined by the normalization condition $\int P_N \Pi_i d\mathbf{r}_i d\mathbf{v}_i=1$. The canonical distribution (7) minimizes the free energy F at fixed particle number. Introducing the reduced probability distributions

$$P_j(\mathbf{x}_1,\ldots,\mathbf{x}_j)=\int P_N(\mathbf{x}_1,\ldots,\mathbf{x}_N)d\mathbf{x}_{j+1}\cdots d\mathbf{x}_N, \quad (8)$$

where $\mathbf{x}=(\mathbf{r},\mathbf{v})$, we can readily write down a hierarchy of equations for P_1, P_2, etc. The first equation of the hierarchy is

$$\frac{\partial P_1}{\partial t}+\mathbf{v}_1\cdot\frac{\partial P_1}{\partial \mathbf{r}_1}+(N-1)\int \mathbf{F}(2\to 1)\cdot\frac{\partial P_2}{\partial \mathbf{v}_1}d\mathbf{r}_2 d\mathbf{v}_2 = \frac{\partial}{\partial \mathbf{v}_1}\cdot\left[D\frac{\partial P_1}{\partial \mathbf{v}_1}+\xi P_1\mathbf{v}_1\right], \quad (9)$$

where $\mathbf{F}(2\to 1)=-m\partial u_{12}/\partial \mathbf{r}_1=Gm(\mathbf{r}_2-\mathbf{r}_1)/|\mathbf{r}_2-\mathbf{r}_1|^d$ is the force (by unit of mass) created by particle 2 on particle 1. Note that this equation is exact (i.e., valid for all N) and takes into account the correlations between the particles encapsulated in the two-body distribution function P_2. For $D=\xi=0$, we recover the first equation of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy. We shall give the form of the virial theorem associated with Eq. (9) in Sec. III C. Before that, we consider the mean-field limit of this equation valid for $N\to+\infty$.

B. Mean-field Kramers equation

In a properly defined thermodynamic limit [12], we can show that the cumulant of the two-body correlation function is of order $1/N$. Therefore, for $N\to+\infty$, we can implement the mean-field approximation

$$P_2(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2,t)=P_1(\mathbf{r}_1,\mathbf{v}_1,t)P_1(\mathbf{r}_2,\mathbf{v}_2,t)+O(1/N). \quad (10)$$

Substituting this result into Eq. (9) and introducing the distribution function $f=NmP_1$, we obtain the mean-field Kramers equation

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}-\nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}}=\frac{\partial}{\partial \mathbf{v}}\cdot\left[D\frac{\partial f}{\partial \mathbf{v}}+\xi f\mathbf{v}\right], \quad (11)$$

where $\Phi(\mathbf{r},t)=\int u(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}',t)d\mathbf{r}'$. This equation is nonlocal because the potential $\Phi(\mathbf{r},t)$ is induced by the density $\rho(\mathbf{r},t)=\int f d\mathbf{v}$ of the particles composing the whole system (it is not an external potential). Thus, for self-gravitating systems, Eq. (11) has to be solved in conjunction with the Poisson equation

$$\Delta\Phi=S_d G\rho. \quad (12)$$

In the absence of friction and diffusion ($D=\xi=0$), the mean-field Kramers equation reduces to the Vlasov equation [1]. The Vlasov equation conserves the energy $E=\frac{1}{2}\int f v^2 d\mathbf{r}d\mathbf{v}+\frac{1}{2}\int\rho\Phi d\mathbf{r}$ and the Boltzmann entropy $S_B=-\int(f/m)\ln(f/m)d\mathbf{r}d\mathbf{v}$ (more generally all the functionals of f called the Casimirs) defined on μ space. Alternatively, the Kramers-Poisson (KP) system (11) and (12) involves a fixed temperature and decreases the Boltzmann free energy $F_B=E-TS_B$ which can be written explicitly

$$F_B[f]=\frac{1}{2}\int f v^2 d\mathbf{r}d\mathbf{v}+\frac{1}{2}\int\rho\Phi d\mathbf{r}+k_B T\int\frac{f}{m}\ln\frac{f}{m}d\mathbf{r}d\mathbf{v}. \quad (13)$$

Indeed, one has

$$\dot{F}_B=-\int\frac{1}{\xi f}\left(D\frac{\partial f}{\partial \mathbf{v}}+\xi f\mathbf{v}\right)^2 d\mathbf{r}d\mathbf{v}\leqslant 0. \quad (14)$$

At equilibrium, $\dot{F}_B=0$, implying that the RHS of Eq. (11) vanishes. The LHS (advective term) must also vanish independently. From these two requirements and using the Einstein relation, we find that the stationary solutions of the Kramers-Poisson system (11) correspond to the mean-field Maxwell-Boltzmann distribution

$$f=Ae^{-\beta m[v^2/2+\Phi(\mathbf{r})]}, \quad (15)$$

which has to be solved in conjunction with the Poisson equation (12). The stable mean-field Maxwell-Boltzmann distribution minimizes the Boltzmann free energy $F_B[f]$ at fixed mass. It is both thermodynamically stable (in the canonical ensemble) and linearly dynamically stable with respect to the KP system [13]. We note that the equilibrium one-body distribution function (15) can be obtained from the N-body canonical distribution (7) by constructing an equilibrium BBGKY-like hierarchy and implementing a mean-field approximation [12]. On the other hand, the Boltzmann free energy (13) can be deduced from the free energy of the N-body system (5) by making the mean-field approximation $P_N(1,\ldots,N)=\Pi_i P_1(i)$ [12].

C. Generalized nonlocal Kramers equation

For sake of generality, we shall consider the case where the diffusion coefficient explicitly depends on the distribution function. Thus, in Eq. (11), we set $D(f)=DfC''(f)$ where C is a convex function—i.e., $C''\geqslant 0$—and D is a constant. In that case, we obtain the generalized mean-field Kramers equation [13]

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}-\nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}}=\frac{\partial}{\partial \mathbf{v}}\cdot\left[DfC''(f)\frac{\partial f}{\partial \mathbf{v}}+\xi f\mathbf{v}\right]. \quad (16)$$

This equation can be obtained in the mean-field limit of a generalized N-body Fokker-Planck equation associated with a generalized class of stochastic processes of the form (1) and (2) where the diffusion coefficient depends on $f(\mathbf{r}_i,\mathbf{v}_i,t)$ [12]. For $C=f\ln f$, we recover the classical Kramers equation (11). However, Eq. (16) can describe more general situations such as quantum particles with exclusion or inclusion

principles (fermions, bosons, quons), lattice models, nonideal effects, etc. These generalized Fokker-Planck equations are associated with an effective thermodynamical formalism (ETF) in μ space [13]. In particular, the generalized Kramers-Poisson (GKP) system decreases the free energy

$$F[f] \equiv E - TS = \int f\frac{v^2}{2}d\mathbf{r}d\mathbf{v} + \frac{1}{2}\int \rho\Phi d\mathbf{r} + T\int C(f)d\mathbf{r}d\mathbf{v}, \tag{17}$$

where the last term can be interpreted as a generalized entropy $S=-\int C(f)d\mathbf{r}d\mathbf{v}$ and we have defined the effective temperature $T=1/\beta$ through the relation $\xi=D/T$ (effective Einstein relation). One has

$$\dot{F} = -\int \frac{1}{\xi f}\left(DfC''(f)\frac{\partial f}{\partial \mathbf{v}} + \xi f\mathbf{v}\right)^2 d\mathbf{r}d\mathbf{v} \leqslant 0. \tag{18}$$

At equilibrium, we find that the stationary solutions of the generalized Kramers equation (16) are determined by the integro-differential equation

$$C'(f) = -\beta\left(\frac{v^2}{2} + \Phi\right) - \alpha, \tag{19}$$

where $\Phi(\mathbf{r},t)=\int u(\mathbf{r}-\mathbf{r}')f(\mathbf{r}',\mathbf{v}',t)d\mathbf{r}'d\mathbf{v}'$. Since C is convex, this relation can be inverted to give $f=F(\beta\epsilon+\alpha)$ where $F(x)=(C')^{-1}(-x)$ and $\epsilon=\frac{v^2}{2}+\Phi(\mathbf{r})$. We note that the equilibrium distribution determined by Eq. (19) is a function $f=f(\epsilon)$ of the individual energy ϵ alone which is monotonically decreasing (for $\beta>0$). This equilibrium distribution function extremizes the free energy (17) at fixed mass. Furthermore, only a (local) minimum of free energy is linearly dynamically stable with respect to the GKP system [13].

D. Damped Jeans equations

We shall now determine the hierarchy of moment equations associated with the generalized Kramers-Poisson system. Defining the density and the local velocity by

$$\rho = \int f d\mathbf{v}, \quad \rho\mathbf{u} = \int f\mathbf{v}d\mathbf{v}, \tag{20}$$

and integrating Eq. (16) on velocity, we get the continuity equation

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0. \tag{21}$$

Next, multiplying Eq. (16) by $\mathbf{v}$ and integrating on velocity, we obtain

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial P_{ij}}{\partial x_j} - \rho\frac{\partial\Phi}{\partial x_i} - \xi\rho u_i, \tag{22}$$

where we have defined the pressure tensor

$$P_{ij} = \int f w_i w_j d\mathbf{v}, \tag{23}$$

where $\mathbf{w}=\mathbf{v}-\mathbf{u}$ is the relative velocity. In the absence of diffusion and friction ($D=\xi=0$), we recover the Jeans equations of astrophysics, which are derived from the Vlasov equation [1]. For self-gravitating Brownian particles, the equivalent of the Jeans equations (21) and (22) includes an additional friction force $-\xi\mathbf{u}$. Using the continuity equation, Eq. (22) can be rewritten

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial P_{ij}}{\partial x_j} - \rho\frac{\partial\Phi}{\partial x_i} - \xi\rho u_i. \tag{24}$$

E. Damped barotropic Euler equations

By taking the successive moments of the velocity, we can obtain a hierarchy of hydrodynamical equations. Each equation of the hierarchy involves the moment of next order. The ordinary Jeans equations that are based on the Vlasov equation are difficult to close because the Vlasov equation admits an infinite number of stationary solutions. Therefore, a notion of thermodynamical equilibrium is difficult to justify in the usual point of view (see, however, [24] in the context of the theory of violent relaxation). In the present case, the situation is simpler because the Kramers equation admits a Lyapunov functional (17) and a unique stationary distribution defined by Eq. (19). If we are sufficiently close to equilibrium, it makes sense to close the hierarchy of equations by using a condition of local thermodynamic equilibrium. We shall thus determine the pressure tensor (23) with the distribution function f_{LTE} defined by the relation

$$C'(f_{LTE}) = -\beta\left[\frac{w^2}{2} + \lambda(\mathbf{r},t)\right]. \tag{25}$$

This distribution function minimizes the generalized free energy (17) at fixed temperature T, local density $\rho(\mathbf{r},t)$, and local velocity $\mathbf{u}(\mathbf{r},t)$. The function $\lambda(\mathbf{r},t)$ is the Lagrange multiplier associated with the density field. It is determined by the requirement

$$\rho(\mathbf{r},t) = \int f_{LTE}d\mathbf{v} = \rho[\lambda(\mathbf{r},t)]. \tag{26}$$

At equilibrium, we recover the distribution function (19) with $\mathbf{u}(\mathbf{r})=\mathbf{0}$ and $\lambda(\mathbf{r})=\Phi(\mathbf{r})+\alpha/\beta$. Using the condition (25) of LTE, the pressure tensor (23) can be written $P_{ij}=p\delta_{ij}$ with

$$p(\mathbf{r},t) = \frac{1}{d}\int f_{LTE}w^2 d\mathbf{w} = p[\lambda(\mathbf{r},t)]. \tag{27}$$

The pressure is a function $p=p(\rho)$ of the density which is entirely specified by the function $C(f)$, by eliminating λ from the relations (26) and (27). We note furthermore that, using Eq. (25) and integrations by parts, the previous equations (26) and (27) easily lead to $\boldsymbol{\nabla}p=\frac{1}{d}\int\frac{\partial f}{\partial\mathbf{r}}w^2 d\mathbf{w}=\frac{1}{d}\boldsymbol{\nabla}\lambda\int\frac{\partial f}{\partial\mathbf{w}}\mathbf{w}d\mathbf{w}=-\boldsymbol{\nabla}\lambda\int f d\mathbf{w}=-\rho\boldsymbol{\nabla}\lambda$; hence, $p'(\rho)=-\rho\lambda'(\rho)$. In the case of Brownian particles described by the ordinary Kramers equation (11) with $C(f)=f\ln f$, the equation of state determined by Eqs. (26) and (27) is the isothermal one $p=\frac{k_BT}{m}\rho$. More generally, we obtain the damped Euler equations for a barotropic gas [13]:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0, \tag{28}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi - \xi\mathbf{u}. \tag{29}$$

These equations decrease the free energy

$$F[\rho,\mathbf{u}] = \int \rho \int^{\rho} \frac{p(\rho')}{\rho'^2} d\rho' d\mathbf{r} + \frac{1}{2}\int \rho\Phi d\mathbf{r} + \int \rho\frac{\mathbf{u}^2}{2} d\mathbf{r}, \tag{30}$$

which can be deduced from the free energy (17) by using the local thermodynamic equilibrium condition (25) to express $F[f]$ as a functional of ρ and $\mathbf{u}$, using $F[\rho,\mathbf{u}]=F[f_{LTE}]$ (see [17] for details). A direct calculation yields

$$\dot{F} = -\xi \int \rho \mathbf{u}^2 d\mathbf{r} \leqslant 0. \tag{31}$$

At equilibrium, $\dot{F}=0$, implying $\mathbf{u}=\mathbf{0}$. Then, Eq. (29) yields the condition of hydrostatic balance:

$$\nabla p + \rho\nabla\Phi = \mathbf{0}, \tag{32}$$

which also results from Eq. (19). Indeed, for $f=f(\epsilon)$ with $\epsilon = v^2/2+\Phi(\mathbf{r})$, one has $\rho=\int f(\epsilon)d\mathbf{v}$, $p=\frac{1}{d}\int f(\epsilon)v^2 d\mathbf{v}$ so that

$$\begin{aligned}\nabla p &= \frac{1}{d}\int f'(\epsilon)\nabla\Phi v^2 d\mathbf{v} \\ &= \frac{1}{d}\nabla\Phi \int \frac{\partial f}{\partial \mathbf{v}}\cdot \mathbf{v} d\mathbf{v} \\ &= -\nabla\Phi \int f d\mathbf{v} \\ &= -\rho\nabla\Phi. \end{aligned} \tag{33}$$

The damped barotropic Euler equations (28) and (29) are interesting as they connect hyperbolic models to parabolic models. Indeed, for $\xi=0$ we recover the standard barotropic Euler equations describing the dynamics of gaseous stars [1,25–27] or the formation of large-scale structures in cosmology [28]. Alternatively, in the strong friction limit $\xi\to+\infty$, we can neglect the inertial term in Eq. (29) and we obtain

$$\mathbf{u} = -\frac{1}{\xi\rho}(\nabla p + \rho\nabla\Phi) + O(\xi^{-2}). \tag{34}$$

Substituting this drift term in the continuity equation (28), we get the generalized Smoluchowski equation [13]

$$\frac{\partial \rho}{\partial t} = \nabla\cdot\left[\frac{1}{\xi}(\nabla p + \rho\nabla\Phi)\right]. \tag{35}$$

This equation decreases the free energy

$$F[\rho] = \int \rho \int^{\rho} \frac{p(\rho')}{\rho'^2} d\rho' d\mathbf{r} + \frac{1}{2}\int \rho\Phi d\mathbf{r}, \tag{36}$$

which is obtained from Eq. (30) by neglecting the last term of order $O(\xi^{-2})$. A direct calculation yields

$$\dot{F} = -\int \frac{1}{\xi\rho}(\nabla p + \rho\nabla\Phi)^2 d\mathbf{r} \leqslant 0. \tag{37}$$

It should be recalled that the damped Euler equations (28) and (29) remain heuristic because their derivation is based on a LTE assumption (25) which is not rigorously justified. However, using a Chapman-Enskog expansion, it is shown in [17] that the generalized Smoluchowski equation (35) is *exact* in the limit $\xi\to+\infty$ (or, equivalently, for times $t\gg\xi^{-1}$). The generalized Smoluchowski equation can also be obtained from the moments equations of the generalized Kramers equation by closing the hierarchy, using $\xi\to+\infty$ (see Sec. 9 of [16]).

F. Orbit-averaged Kramers equation

We shall consider here the opposite limit of low friction, $\xi\to 0$. In the case where the term on the RHS of Eq. (11) is small, we can obtain a simplified equation for the evolution of the distribution function by averaging the kinetic equation over the orbits of the particles. In the case of Hamiltonian self-gravitating systems described by the Landau equation, this leads to the orbit-averaged Landau equation introduced by Hénon [29]. We shall here derive the orbit-averaged Kramers equation for self-gravitating Brownian systems.

Let us first rewrite the mean-field Kramers equation (11) in the form

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} = Q(f). \tag{38}$$

We consider the case where $\xi\to 0$ for fixed β so that $\xi\simeq D \simeq 0$ and the operator $Q(f)$ can be considered as small. If we take $Q(f)=0$, we obtain the Vlasov equation. We shall assume that the system has reached a stable stationary distribution of the Vlasov equation of the form $f=f(\epsilon)$ which depends only on the energy $\epsilon=\frac{v^2}{2}+\Phi(\mathbf{r},t)$ of the particles. This is a particular case of the Jeans theorem for spherical systems. Such a steady solution can arise from a process of violent collisionless relaxation [30,12]. Since $Q(f)\neq 0$, the distribution function will change due to the terms of friction and diffusion that are present in the stochastic equation Eq. (2).[2] However, if $\xi\to 0$, this change will be slow so that the latter forces cause only a small variation on the energy. We shall therefore consider that $f(\mathbf{r},\mathbf{v},t)\simeq f(\epsilon,t)$ remains a function of the energy alone that slowly evolves due to imposed stochastic forces. Noting that

$$\frac{\partial}{\partial t} f(\epsilon(\mathbf{r},\mathbf{v},t),t) = \frac{\partial f}{\partial t} + \frac{\partial\Phi}{\partial t}\frac{\partial f}{\partial \epsilon}, \tag{39}$$

we can rewrite the kinetic equation (38) in the form

[2] In the case of Hamiltonian systems where $\xi=D=0$, the distribution function changes due to finite-N effects representing close encounters between stars. These encounters are usually modeled by the Landau operator [12] which is of order $1/N\ll 1$. On the other hand, for self-gravitating Brownian particles the evolution is modeled by the Kramers operator (obtained for $N\to+\infty$) which is of the order ξt_D (where t_D is the dynamical time). The comparison between the evolution of Hamiltonian and Brownian systems with long-range interactions is further discussed in the Conclusion of this paper.

$$\frac{\partial f}{\partial t}+\frac{\partial \Phi}{\partial t}\frac{\partial f}{\partial \epsilon}=Q(f). \tag{40}$$

Since f depends only on the energy, the system is spherically symmetric. Then, the phase space hypersurface with energy smaller than ϵ is

$$q(\epsilon,t)\equiv 16\pi^2\int_{v^2/2+\Phi\leqslant\epsilon} r^2dr v^2dv$$
$$=\frac{16\pi^2}{3}\int_0^{r_{max}(\epsilon,t)}[2(\epsilon-\Phi)]^{3/2}r^2dr, \tag{41}$$

where $r_{max}(\epsilon,t)$ is the largest radial extent of an orbit with energy ϵ at time t. It is determined by the equation $\epsilon=\Phi(r_{max},t)$ corresponding to $v=0$. The previous relation can be written more compactly as

$$q(\epsilon,t)=\frac{16\pi^2}{3}\int_0^{r_{max}} v^3r^2dr, \tag{42}$$

where $v=\sqrt{2[\epsilon-\Phi(r,t)]}$. The phase-space hypersurface $g(\epsilon,t)d\epsilon$ with energy between ϵ and $\epsilon+d\epsilon$ is given by

$$g(\epsilon,t)\equiv\frac{\partial q}{\partial\epsilon}=16\pi^2\int_0^{r_{max}(\epsilon,t)}[2(\epsilon-\Phi)]^{1/2}r^2dr$$
$$=16\pi^2\int_0^{r_{max}} vr^2dr. \tag{43}$$

Now, the density of particles in the hypersurface between ϵ and $\epsilon+d\epsilon$ is uniform since the distribution function depends only on the energy. In fact, the system evolves on a short time scale $\sim t_D$ by purely inertial effects [corresponding to the advective terms in the LHS of Eq. (38)] so as to establish this quasistationary regime where $f\simeq f(\epsilon,t)$. We shall therefore average the kinetic equation (40) on each hypersurface of isoenergy using

$$\langle X\rangle(\epsilon,t)=\frac{\int_0^{r_{max}} Xvr^2dr}{\int_0^{r_{max}} vr^2dr} \tag{44}$$

for any function $X(r,v,t)$. Thus, the orbit-averaged kinetic equation can be written

$$16\pi^2\int_0^{r_{max}} r^2dr\, v\left[\frac{\partial f}{\partial t}+\frac{\partial\Phi}{\partial t}\frac{\partial f}{\partial\epsilon}-Q(f)\right]=0. \tag{45}$$

The first term in the brackets is

$$16\pi^2\int_0^{r_{max}} r^2dr\, v\frac{\partial f}{\partial t}=g(\epsilon,t)\frac{\partial f}{\partial t}=\frac{\partial q}{\partial\epsilon}\frac{\partial f}{\partial t}. \tag{46}$$

Since

$$\frac{\partial q}{\partial t}=-16\pi^2\int_0^{r_{max}} v\frac{\partial\Phi}{\partial t}r^2dr, \tag{47}$$

the second term in the brackets can be written

$$16\pi^2\int_0^{r_{max}} r^2dr\, v\frac{\partial\Phi}{\partial t}\frac{\partial f}{\partial\epsilon}=-\frac{\partial q}{\partial t}\frac{\partial f}{\partial\epsilon}. \tag{48}$$

Finally, since

$$vQ(f)=D\frac{\partial}{\partial\epsilon}\left[v^3\left(\frac{\partial f}{\partial\epsilon}+\beta f\right)\right], \tag{49}$$

the last term in the brackets is

$$16\pi^2\int_0^{r_{max}} r^2dr\, vQ(f)=3D\frac{\partial}{\partial\epsilon}\left[q\left(\frac{\partial f}{\partial\epsilon}+\beta mf\right)\right]. \tag{50}$$

Regrouping all these results, we obtain the orbit-averaged Kramers equation

$$\frac{\partial q}{\partial\epsilon}\frac{\partial f}{\partial t}-\frac{\partial q}{\partial t}\frac{\partial f}{\partial\epsilon}=3D\frac{\partial}{\partial\epsilon}\left[q\left(\frac{\partial f}{\partial\epsilon}+\beta mf\right)\right], \tag{51}$$

$$q(\epsilon,t)=\frac{16\pi^2}{3}\int_0^{r_{max}(\epsilon,t)}\{2[\epsilon-\Phi(r,t)]\}^{3/2}r^2dr, \tag{52}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)=16\pi^2G\int_{\Phi(r,t)}^{+\infty} f(\epsilon,t)[2(\epsilon-\Phi(r,t))]^{1/2}r^2d\epsilon, \tag{53}$$

where the last equation is the Poisson equation. It is easy to verify that the free energy is monotonically decreasing ($\dot F\leqslant 0$) and that the stationary solution of Eq. (51) is the Boltzmann distribution $f=Ae^{-\beta m\epsilon}$. These equations will be studied in a future communication. Note also that in $d=1$, $q=\oint v dx$, and $g=2\oint dx/v$, and in $d=2$, $q=2\pi^2\int_0^{r_m}v^2rdr$, and $g=2\pi^2r_m^2$.

III. VIRIAL THEOREM FOR BROWNIAN PARTICLES

A. Virial theorem from the damped Jeans equations

We shall give here the form of the virial theorem appropriate to the damped Jeans equations (21) and (22). The only difference with the standard Jeans equations is the presence of a friction term. We shall thus only give the final result and refer to [1] for the details of calculation. The damped virial theorem can be written

$$\frac{1}{2}\frac{d^2I_{ij}}{dt^2}+\frac{1}{2}\xi\frac{dI_{ij}}{dt}=2K_{ij}+W_{ij}-\frac{1}{2}\oint(P_{ik}x_j+P_{jk}x_i)dS_k. \tag{54}$$

We have included boundary terms which must be taken into account if the system is confined within a box. The tensor of inertia I_{ij} and the potential energy tensor W_{ij} are defined in paper I. The kinetic energy tensor is defined by

$$K_{ij}=\frac{1}{2}\int f v_iv_j d\mathbf{v}. \tag{55}$$

It can be written as

$$K_{ij} = T_{ij} + \frac{1}{2}\Pi_{ij}, \tag{56}$$

where

$$T_{ij} = \frac{1}{2}\int \rho u_i u_j d\mathbf{r}, \quad \Pi_{ij} = \int P_{ij} d\mathbf{r}. \tag{57}$$

Note that the tensors K_{ij} and P_{ij} depend on the distribution function $f(\mathbf{r},\mathbf{v},t)$, not only on hydrodynamic variables. The scalar virial theorem takes the form

$$\frac{1}{2}\frac{d^2I}{dt^2} + \frac{1}{2}\xi\frac{dI}{dt} = 2K + W_{ii} - \oint P_{ik}x_i dS_k, \tag{58}$$

where I is the moment of inertia and

$$K = \frac{1}{2}\int f v^2 d\mathbf{r}d\mathbf{v} \tag{59}$$

is the kinetic energy. It can be written

$$K = T + \frac{1}{2}\Pi, \tag{60}$$

where

$$T = \frac{1}{2}\int \rho \mathbf{u}^2 d\mathbf{r}, \quad \Pi = d\int p d\mathbf{r}. \tag{61}$$

In the absence of diffusion and friction ($D=\xi=0$), we recover the usual expression of the virial theorem issued from the Jeans equations [1]. For Brownian particles, the novelty is the presence of a damping term $\xi\dot{I}$.

As pointed out in paper I, the moment of inertia depends on the origin O of the system of coordinates. Let $\mathbf{R}(t) = (1/M)\int \rho \mathbf{r} d\mathbf{r}$ denote the position of the center of mass with respect to the origin O. Using the equation of continuity (28), we find that $Md\mathbf{R}/dt = \mathbf{P}$ where $\mathbf{P} = \int \rho \mathbf{u} d\mathbf{r}$ is the total impulse. Using the Jeans equation (22), we find that $d\mathbf{P}/dt = -\xi\mathbf{P}$. In our case, there exists an absolute referential ($\mathcal{R}$). Indeed, in writing Eq. (2) we have implicitly assumed that our Brownian particles evolve in a fluid that is motionless. Otherwise, the friction force in Eq. (2) has to be modified according to $-\xi(\mathbf{v}_i - \mathbf{U})$ where $\mathbf{U}$ is the velocity of the fluid [17]. We must work therefore in this referential ($\mathcal{R}$). If we denote by $\mathbf{P}_0$ the initial value of the impulse, we get $\mathbf{P}(t) = \mathbf{P}_0 e^{-\xi t}$. If now $\mathbf{R}_0$ denotes the initial position of the center of mass with respect to O, we find that

$$\mathbf{R}(t) = \mathbf{R}_0 + \frac{\mathbf{P}_0}{M\xi}(1 - e^{-\xi t}). \tag{62}$$

Therefore, at $t \to +\infty$ the center of mass has been shifted by a quantity $\mathbf{P}_0/M\xi$. In the strong friction limit $\xi \to +\infty$, we find that the center of mass is motionless (paper I).

B. Virial theorem from the damped Euler equations

The virial theorem associated with the damped barotropic Euler equations (28) and (29) can be deduced from Eq. (54) by using the fact that $P_{ij} = p(\rho)\delta_{ij}$. This yields

$$\frac{1}{2}\frac{d^2I_{ij}}{dt^2} + \frac{1}{2}\xi\frac{dI_{ij}}{dt} = 2T_{ij} + \frac{1}{d}\Pi\delta_{ij} + W_{ij} - \frac{1}{2}\oint p(\delta_{ik}x_j + \delta_{jk}x_i)dS_k, \tag{63}$$

$$\frac{1}{2}\frac{d^2I}{dt^2} + \frac{1}{2}\xi\frac{dI}{dt} = 2T + \Pi + W_{ii} - \oint p\mathbf{r}\cdot d\mathbf{S}, \tag{64}$$

where each quantity is now expressed in terms of hydrodynamic variables. At equilibrium, if no macroscopic motion is present ($T=0$) and if we can neglect the boundary term, we get

$$W_{ij} = -\frac{1}{d}\Pi\delta_{ij}. \tag{65}$$

In the absence of diffusion and friction ($D=\xi=0$), we recover the usual form of the virial theorem issued from the barotropic Euler equations [1]. Alternatively, in the strong-friction limit $\xi \to +\infty$, we can neglect the term $\ddot{I}$ in front of $\xi\dot{I}$. Furthermore, since the velocity field scales as $\mathbf{u} = O(\xi^{-1})$, the kinetic energy tensor $T_{ij} = O(\xi^{-2})$ can also be neglected. Therefore, the virial theorem associated with the generalized Smoluchowski-Poisson system (35) and (12) can be written

$$\frac{1}{2}\xi\frac{dI_{ij}}{dt} = 2K_{ij} + W_{ij} - \frac{1}{2}\oint p(\delta_{ik}x_j + \delta_{jk}x_i)dS_k, \tag{66}$$

$$\frac{1}{2}\xi\frac{dI}{dt} = 2K + W_{ii} - \oint p\mathbf{r}\cdot d\mathbf{S}, \tag{67}$$

where

$$K_{ij} = \frac{1}{d}K\delta_{ij}, \quad K = \frac{d}{2}\int p d\mathbf{r}, \tag{68}$$

is the expression of the kinetic energy to leading order in the limit $\xi \to +\infty$ where $\mathbf{u}(\mathbf{r},t)$ can be neglected. We thus recover the results of paper I starting directly from the GSP system.

C. Effect of correlations

If we account for the effect of correlations (due to finite values of N) between the particles and use the exact kinetic equation (9), we obtain the exact damped Jeans equations

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0, \tag{69}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial P_{ij}}{\partial x_j} + Gm^2\int \frac{x_i' - x_i}{|\mathbf{r}' - \mathbf{r}|^d}\rho_2(\mathbf{r},\mathbf{r}',t)d\mathbf{r}' - \xi\rho u_i, \tag{70}$$

where we have introduced the spatial correlation function

$$\rho_2(\mathbf{r}_1,\mathbf{r}_2,t) = N(N-1)\int P_2(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2,t)d\mathbf{v}_1 d\mathbf{v}_2. \tag{71}$$

In the mean-field approximation $\rho_2(1,2) = \rho(1)\rho(2)$, we recover the damped Jeans equations (21) and (22). The virial theorem is now given by

$$\frac{1}{2}\frac{d^2I_{ij}}{dt^2}+\frac{1}{2}\xi\frac{dI_{ij}}{dt}=2K_{ij}+W_{ij}^{corr}-\frac{1}{2}\oint(P_{ik}x_j+P_{jk}x_i)dS_k, \tag{72}$$

where

$$W_{ij}^{corr}=-\frac{Gm^2}{2}\int\rho_2(\mathbf{r},\mathbf{r}')\frac{(x_i-x_i')(x_j-x_j')}{|\mathbf{r}-\mathbf{r}'|^d}d\mathbf{r}d\mathbf{r}' \tag{73}$$

is a generalization of the potential energy tensor accounting for correlations between particles. In the strong-friction limit $\xi\to+\infty$, the virial theorem reduces to

$$\frac{1}{2}\xi\frac{dI_{ij}}{dt}=\delta_{ij}\int pd\mathbf{r}+W_{ij}^{corr}-\frac{1}{2}\oint p(\delta_{ik}x_j+\delta_{jk}x_i)dS_k. \tag{74}$$

If we consider the case of Brownian particles with an isothermal equation of state $p=\rho k_BT/m$ and if we focus on a space with $d=2$ dimensions where

$$W_{ii}^{corr}=-\frac{Gm^2}{2}\int\rho_2(\mathbf{r},\mathbf{r}')d\mathbf{r}d\mathbf{r}'=-N(N-1)\frac{Gm^2}{2}, \tag{75}$$

the scalar virial theorem takes the form

$$\frac{1}{2}\xi\frac{dI}{dt}=2Nk_B(T-T_c)-2PV, \tag{76}$$

with a critical temperature

$$k_BT_c=\frac{Gm^2(N-1)}{4}. \tag{77}$$

These results are valid for an arbitrary number of particles. For $N\gg1$, using $N-1\simeq N$, we recover the results of paper I.

IV. DYNAMICAL STABILITY OF SELF-GRAVITATING BROWNIAN PARTICLES

We shall now investigate the linear dynamical stability of a stationary solution of the damped barotropic Euler-Poisson system (28) and (29) satisfying the condition of hydrostatic balance (32). We shall determine in particular the equation of pulsations satisfied by a small perturbation around this equilibrium state. We shall investigate the influence of the friction parameter ξ on the pulsation period and make the connection between the standard Euler-Poisson system (hyperbolic) obtained for $\xi=0$ and the generalized Smoluchowski-Poisson system (parabolic) obtained for $\xi\to+\infty$. The linearized damped Euler-Poisson equations are

$$\frac{\partial\delta\rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\delta\mathbf{u})=0, \tag{78}$$

$$\rho\frac{\partial\delta\mathbf{u}}{\partial t}=-\boldsymbol{\nabla}(p'(\rho)\delta\rho)-\rho\boldsymbol{\nabla}\delta\Phi-\delta\rho\boldsymbol{\nabla}\Phi-\xi\rho\delta\mathbf{u}, \tag{79}$$

$$\Delta\delta\Phi=S_dG\delta\rho. \tag{80}$$

Considering spherically symmetric systems and writing the evolution of the perturbation as $\delta\rho\sim e^{\lambda t}$, we get

$$\lambda\delta\rho+\frac{1}{r^{d-1}}\frac{d}{dr}(r^{d-1}\rho\delta u)=0, \tag{81}$$

$$\lambda\rho\delta u=-\frac{d}{dr}(p'(\rho)\delta\rho)-\rho\frac{d\delta\Phi}{dr}-\delta\rho\frac{d\Phi}{dr}-\xi\rho\delta u, \tag{82}$$

$$\frac{1}{r^{d-1}}\frac{d}{dr}\left(r^{d-1}\frac{d\delta\Phi}{dr}\right)=S_dG\delta\rho. \tag{83}$$

As in paper I, we introduce the function $q(r)$ defined by

$$\delta\rho=\frac{1}{S_dr^{d-1}}\frac{dq}{dr}. \tag{84}$$

The continuity equation then yields

$$\delta u=-\frac{\lambda q}{S_d\rho r^{d-1}}. \tag{85}$$

After some elementary transformations similar to those of paper I, Eq. (82) can be put in the form

$$\frac{d}{dr}\left(\frac{p'(\rho)}{S_d\rho r^{d-1}}\frac{dq}{dr}\right)+\frac{Gq}{r^{d-1}}=\frac{\lambda(\lambda+\xi)}{S_d\rho r^{d-1}}q. \tag{86}$$

The case of barotropic stars described by the Euler-Poisson system corresponds to $\xi=0$ [1,25–27]. The case of self-gravitating Brownian particles described by the generalized Smoluchowski-Poisson system is recovered for $\xi\gg\lambda$ (see [13] and paper I). We can therefore use the results of paper I by making the substitution $\xi\lambda\to\lambda(\lambda+\xi)$. Therefore, an approximate analytical expression for the eigenvalue λ is given by

$$\lambda(\lambda+\xi)=(d\overline{\gamma}+2-2d)(d-2)\frac{W}{I}\quad(d\neq2), \tag{87}$$

$$\lambda(\lambda+\xi)=-(\overline{\gamma}-1)\frac{GM^2}{I}\quad(d=2). \tag{88}$$

The friction coefficient ξ affects the evolution of the instability but it does not change the instability threshold [determined by the sign of the LHS of Eqs. (87) and (88)]. The unstable case corresponds to $\lambda(\lambda+\xi)=\sigma^2>0$. The two eigenvalues are

$$\lambda_\pm=\frac{-\xi\pm\sqrt{\xi^2+4\sigma^2}}{2}. \tag{89}$$

Since $\lambda_+>0$, we see that the perturbation grows exponentially rapidly as e^{λ_+t}. The stable case corresponds to $\lambda(\lambda+\xi)=-\sigma^2<0$. The two eigenvalues are

$$\lambda_\pm=\frac{-\xi\pm\sqrt{\xi^2-4\sigma^2}}{2}. \tag{90}$$

If $\xi^2-4\sigma^2\geqslant0$, then $\lambda_\pm<0$ and the perturbation decreases exponentially rapidly without oscillating. This is the case in

particular for Brownian particles described by the Smoluchowski equation ($\xi \to +\infty$) for which $\lambda=-\sigma^2/\xi$ (paper I). Alternatively, if $\xi^2-4\sigma^2 \leqslant 0$, then $\lambda_\pm=(-\xi \pm i\sqrt{4\sigma^2-\xi^2})/2$ and we have slowly damped oscillations with a pulsation $\omega=\frac{1}{2}\sqrt{4\sigma^2-\xi^2}$ and a damping rate $\xi/2$. This is the case in particular for barotropic stars ($\xi=0$) which oscillate with pulsation $\omega=\sigma$ without attenuation. The separation between these two regimes (pure damping versus damped oscillations) is obtained for $\xi=2\sigma$ at which $\omega=0$. This suggests introducing the dimensionless parameter

$$F \equiv \frac{\xi^2}{\lambda(\lambda+\xi)}, \tag{91}$$

measuring the efficiency of the friction force. The critical values are $F=0$ and $F=-4$. If $F<-4$, the system is stable and a perturbation is damped out exponentially rapidly without oscillating. If $-4<F<0$, the system is stable and a perturbation exhibits damped oscillations. The pulsation vanishes for $F=-4$, and the damping rate vanishes for $F=0$. For $F>0$, the system is unstable. Using Eqs. (87) and (88), the parameter defined in Eq. (91) is explicitly given by

$$F=\frac{1}{(3\bar{\gamma}-4)}\frac{\xi^2 I}{W} \quad (d=3), \tag{92}$$

$$F=\frac{-1}{\bar{\gamma}-1}\frac{\xi^2 I}{GM^2} \quad (d=2), \tag{93}$$

$$F=-\frac{1}{\bar{\gamma}}\frac{\xi^2 I}{W}, \quad (d=1). \tag{94}$$

Dimensionally, this parameter scales as $|F| \sim \xi^2 R^d/GM$. It can also be written $|F| \sim (\xi t_D)^2$ where $t_D \sim 1/\sqrt{\rho G}$ is the dynamical time [1]. The dynamical stability of a homogeneous system (for a general form of potential of interaction) is treated in Appendix C.

V. CONCLUSION

In this paper, we have introduced general models of self-gravitating Brownian particles (stochastic N body, kinetic, hydrodynamic, etc.) that relax the simplifying assumptions that are usually considered: mean-field approximation for $N \to +\infty$ (thermodynamic limit) and overdamped approximation for $\xi \to +\infty$ (strong friction limit). These general models show the connection between previously considered models and offer a unifying framework to study these systems. We have focused here on the case of self-gravitating systems but most of our results also apply to the problem of chemotaxis in biology. This will be specifically considered in another paper where we discuss inertial models of bacterial populations.

It should be emphasized that the Brownian model (1) and (2) contains the standard Hamiltonian model of stellar dynamics [1] as a special case since the Langevin equations reduce to the Hamilton equations for $\xi=D=0$. We expect therefore to have different regimes depending on the value of the parameters. To characterize these regimes properly, it is useful to introduce different time scales: (i) The *dynamical time* $t_D \sim 1/\sqrt{\rho G}$ (Kepler time) is the typical period of an orbit or a typical free-fall time [1]. (ii) The *collisional relaxation time* $t_R \sim (N/\ln N)t_D$ (Chandrasekhar time) is the typical time it takes a stellar system (Hamiltonian) to relax to the Boltzmann distribution $e^{-\beta m\epsilon}$ due to close encounters. This relaxation is due to finite-N effects [31]. (iii) The *friction time* $t_B \sim \xi^{-1}$ (Kramers time) is the typical time it takes a Brownian system to thermalize—i.e., to have its velocity distribution close to the Maxwellian $e^{-\beta m v^2/2}$ [32]. This thermalization is due to the combined effect of imposed friction and diffusion in the Langevin model (1) and (2). It is due to a thermal bath (of nongravitational origin), *not* to collisions (finite-N effects). We can now distinguish different cases.

(i) The case $t_D \ll t_R \ll t_B$ ($\xi \to 0$) corresponds to Hamiltonian systems. For $t \ll t_R$, the system is described by the Vlasov-Poisson system. There is first a phase of *violent collisionless relaxation* on a time scale $\sim t_D$ leading to a quasistationary state (QSS) in mechanical equilibrium. This is a stable stationary solution of the Vlasov equation (on the coarse-grained scale), which is usually not described by the Boltzmann distribution. On a longer time scale $t_R \sim (N/\ln N)t_D$ the encounters between stars (due to finite-N effects) have the tendency to drive the system towards a statistical equilibrium state described by the Boltzmann distribution. In reality, this process is hampered by the escape of stars and the gravothermal catastrophe. The collisional evolution of the system is described by the Landau-Poisson system which is the $1/N$ correction to the Vlasov limit (it singles out the Boltzmann distribution among all stationary solutions of the Vlasov equation) [12]. In fact, due to the time-scale separation between the phase of violent relaxation (inertial effects) and the phase of collisional relaxation (finite-N effects), we can consider for intermediate times that the distribution function is a quasistationary solution of the Vlasov equation of the form $f=f(\epsilon,t)$ (for spherical systems) that slowly evolves under the action of close encounters according to the orbit-averaged Landau equation (traditionally called orbit-averaged Fokker-Planck equation). This implies that the lifetime of the QSS is *long* as it increases as a power of N. It *slowly* evolves under the effect of encounters which act as a perturbation of order $1/N$ with $N \gg 1$. Therefore, the system first reaches a state of mechanical equilibrium (through violent relaxation), then a state of thermal equilibrium (through stellar encounters). These different phases of the dynamical evolution of Hamiltonian stellar systems have been studied by astrophysicists for a long time [1].

(ii) The case $t_B \ll t_D \ll t_R$ ($\xi \to +\infty$) corresponds to the overdamped limit of the Brownian model. The velocities first relax towards the Maxwellian distribution on a time scale $t_B \sim \xi^{-1}$ (due to the thermal bath), and the density relaxes towards a state of mechanical equilibrium on a longer time scale (Smoluchowski diffusive time). Therefore, the system first reaches a state of thermal equilibrium (because of the terms of friction and noise in the Langevin equations), then a state of mechanical equilibrium (through inertial effects). This overdamped regime, described by the Smoluchowski-Poisson system, has been studied in our series of papers [4–11].

(iii) Finally, there is an interesting case $t_D \ll t_B \ll t_R$, which has not yet been studied. In that case, there is first a phase of violent relaxation on a time scale $\sim t_D$ leading to a QSS in mechanical equilibrium like in case (i). This phase is followed by a thermalization leading to the Boltzmann distribution on a time scale $t_B \sim \xi^{-1}$ due to the thermal bath—i.e., the combined effect of imposed friction and diffusion in the Langevin model (1) and (2)—*not* to "collisions" (finite-N effects) as in case (i). The first phase is described by the Vlasov-Poisson system and the second phase by the Kramers-Poisson system. For $\xi \to 0$ [but $\xi \gg (\ln N/N) t_D^{-1}$] there is a time-scale separation between the phase of violent relaxation and the phase of Brownian relaxation. Similarly to case (i), we can consider for intermediate times that the distribution function is a quasistationary solution of the Vlasov equation of the form $f=f(\epsilon,t)$ (for spherical systems) that slowly evolves under the action of imposed friction and diffusion (thermal bath, not collisions) according to the orbit-averaged Kramers equation derived in Sec. II F. Since the Brownian time scale $t_B \sim \xi^{-1}$ is independent of N, this implies that the lifetime of the QSS in this regime is independent of N. Furthermore, it is *shorter* than in case (i) if $\xi \gg (\ln N/N) t_D^{-1}$. Therefore, the system first reaches a state of mechanical equilibrium (through violent relaxation), then a state of thermal equilibrium (through the effect of imposed fluctuation and dissipation—i.e., the thermal bath). This is the opposite situation to case (ii). The study of the orbit-averaged Kramers equation will be considered in a future work. Note that if t_B and t_R are comparable, one must take into account simultaneously the effect of the thermal bath (friction and random force) and the effect of collisions (finite-N effects). This is another interesting case. Finally, we stress that these different regimes should be observed for other potentials of interaction $u(|\mathbf{r}-\mathbf{r}'|)$ than the gravitational one (e.g., for the HMF and BMF models [33]). Kinetic theories of Hamiltonian and Brownian particles with long-range interactions are discussed in [12] at a general level.

APPENDIX A: EIGENVALUE EQUATION FOR POLYTROPES

In the case of a polytropic equation of state, we can put the eigenvalue equation (86) in a dimensionless form [26]

$$\frac{d}{d\xi}\left(\frac{\theta^{1-n}}{\xi^{d-1}}\frac{dq}{d\xi}\right) + \frac{nq}{\xi^{d-1}} = \frac{n\Omega^2}{\theta^n \xi^{d-1}} q, \tag{A1}$$

where $\theta(\xi)$ in the Emden function [34] and

$$\Omega^2 = \frac{\lambda(\lambda+\xi)}{S_d G \rho_0} \tag{A2}$$

is the dimensionless eigenvalue. This relation shows that the dimensional eigenvalue scales as $\lambda(\lambda+\xi) \propto G\rho_0$ where ρ_0 is the central density. Now, for a polytrope, the central density is connected to the mean density $\bar{\rho}$ by a proportionality constant depending only on the polytropic index n [34]. Therefore, we also have $\lambda(\lambda+\xi) \propto G\bar{\rho}$.

APPENDIX B: THE EXACT VIRIAL THEOREM IN d DIMENSIONS

We consider N Brownian particles with mass m_α in gravitational interaction. Their equations of motion are

$$\ddot{x}_i^\alpha = \sum_{\beta\neq\alpha} \frac{G m_\beta (x_i^\beta - x_i^\alpha)}{|\mathbf{r}_\beta - \mathbf{r}_\alpha|^d} - \xi \dot{x}_i^\alpha + \sqrt{2D_\alpha} R_i^\alpha(t). \tag{B1}$$

Here, the Greek letters refer to the particles and the Latin letters to the coordinates of space. For simplicity, we have assumed that the friction coefficient ξ is the same for all the particles but this assumption can be relaxed easily (see below). The diffusion coefficient is given by the Einstein formula $D_\alpha = \xi k_B T/m_\alpha$. The multispecies Smoluchowski-Poisson system has been studied in [11]. In this appendix, we establish the exact virial theorem associated with the stochastic equations (B1). The moment of inertia tensor is defined by

$$I_{ij} = \sum_\alpha m_\alpha x_i^\alpha x_j^\alpha. \tag{B2}$$

We introduce the kinetic energy tensor

$$K_{ij} = \frac{1}{2}\sum_\alpha m_\alpha \dot{x}_i^\alpha \dot{x}_j^\alpha \tag{B3}$$

and the potential energy tensor

$$W_{ij} = G\sum_{\alpha\neq\beta} m_\alpha m_\beta \frac{x_i^\alpha (x_j^\beta - x_j^\alpha)}{|\mathbf{r}_\beta - \mathbf{r}_\alpha|^d} = -\frac{1}{2} G \sum_{\alpha\neq\beta} m_\alpha m_\beta \frac{(x_i^\alpha - x_i^\beta)(x_j^\alpha - x_j^\beta)}{|\mathbf{r}_\beta - \mathbf{r}_\alpha|^d}, \tag{B4}$$

where the second equality results from simple algebraic manipulations obtained by exchanging the dummy variables α and β. Taking the second derivative of Eq. (B2), using the equation of motion (B1), and averaging over the noise and on statistical realizations, we get

$$\frac{1}{2}\langle \ddot{I}_{ij}\rangle + \frac{1}{2}\xi\langle \dot{I}_{ij}\rangle = 2\langle K_{ij}\rangle + \langle W_{ij}\rangle - \frac{1}{2}\oint (P_{ik}x_j + P_{jk}x_i) dS_k, \tag{B5}$$

where we have included the virial of the pressure force $F_i = P_{ik}\Delta S_k$ on the boundary of a box: $(\ddot{I}_{ij})_{box} = \Sigma_{box}\Sigma_\alpha (F_i^\alpha x_j^\alpha + F_j^\alpha x_i^\alpha) = \Sigma_{box}\Sigma_\alpha (P_{ik}^\alpha x_j^\alpha + P_{jk}^\alpha x_i^\alpha)\Delta S_k$.[3] The scalar virial theorem is obtained by contracting the indices

$$\frac{1}{2}\langle \ddot{I}\rangle + \frac{1}{2}\xi\langle \dot{I}\rangle = 2\langle K\rangle + \langle W_{ii}\rangle - \oint P_{ik}x_i dS_k, \tag{B6}$$

where

[3]Note that if the particles have a different friction parameter ξ_α, the term $\xi\dot{I}$ is replaced by $\Sigma_\alpha \xi_\alpha m_\alpha (\dot{x}_i^\alpha x_j^\alpha + x_i^\alpha \dot{x}_j^\alpha)$ which can also be written $\Sigma_s \xi_s \dot{I}_{ij,s}$ where (s) denotes the different species of particles and $I_{ij,s}$ is the moment of inertia due to particles of species s.

$$I = \sum_\alpha m_\alpha r_\alpha^2, \quad K = \frac{1}{2}\sum_\alpha m_\alpha v_\alpha^2, \tag{B7}$$

are the moment of inertia and the kinetic energy. On the other hand,

$$W_{ii} = -\frac{1}{2}G\sum_{\alpha\neq\beta}\frac{m_\alpha m_\beta}{|\mathbf{r}_\beta - \mathbf{r}_\alpha|^{d-2}}. \tag{B8}$$

For $d\neq 2$, we find that

$$W_{ii} = (d-2)W, \tag{B9}$$

where W is the potential energy:

$$W = -\frac{G}{2(d-2)}\sum_{\alpha\neq\beta}\frac{m_\alpha m_\beta}{|\mathbf{r}_\beta - \mathbf{r}_\alpha|^{d-2}}. \tag{B10}$$

In that case, the scalar virial theorem reads

$$\frac{1}{2}\langle\ddot{I}\rangle + \frac{1}{2}\xi\langle\dot{I}\rangle = 2\langle K\rangle + (d-2)\langle W\rangle - \oint P_{ik}x_i dS_k. \tag{B11}$$

For Hamiltonian systems ($D=\xi=0$), the total energy $E=K+W$ is conserved. Hence, the virial theorem can be written in an unbounded domain ($P=0$):

$$\frac{1}{2}\ddot{I} = 2K + (d-2)W = 2E + (d-4)W. \tag{B12}$$

This is the extension of the usual virial theorem in d dimensions (this equation is exact without averages). We note that the dimension $d=4$ is critical as also noticed in [35] using different arguments. In that case, $\ddot{I}=4E$ which yields after integration $I=2Et^2+C_1t+C_2$. For $E>0$, $I\to+\infty$, indicating that the system evaporates. For $E<0$, I goes to zero in a finite time, indicating that the system forms a Dirac peak in a finite time. More generally, for $d\geqslant 4$, since $(d-4)W\leqslant 0$, we have $I\leqslant 2Et^2+C_1t+C_2$ so that the system forms a Dirac peak in a finite time if $E<0$ (this remains true for a box-confined system). Therefore, self-gravitating systems with $E<0$ are not stable in a space of dimension $d\geqslant 4$.The study in Ref. [35] indicates that this observation remains true even if quantum effects (Pauli exclusion principle) are taken into account. This is a striking result because quantum mechanics stabilizes matter against gravitational collapse in $d<4$ [36]. For $2<d\leqslant 4$, since $(d-4)W\geqslant 0$ we conclude, according to Eq. (B12), that the system evaporates if $E>0$ while an equilibrium is possible (but not compulsory) if $E<0$. Finally, for $d<2$, since $W>0$, the energy is necessary positive ($E>0$). Since $(d-4)W<0$ an equilibrium state is possible.

For $d=2$, we have the simple result

$$W_{ii} = -\frac{1}{2}G\sum_{\alpha\neq\beta} m_\alpha m_\beta. \tag{B13}$$

For equal-mass particles,

$$W_{ii} = -\frac{1}{2}GN(N-1)m^2, \tag{B14}$$

which reduces to

$$W_{ii} = -\frac{GM^2}{2}, \tag{B15}$$

for $N\gg 1$. We also note that

$$\sum_{\alpha\neq\beta} m_\alpha m_\beta = M^2 - \sum_\alpha m_\alpha^2. \tag{B16}$$

The first term is of order N^2m^2 and the second of order Nm^2 (where m is a typical mass). Therefore, in the mean-field limit $N\to+\infty$, we recover Eq. (B15) even for a multicomponents system; cf [11].

At equilibrium, the virial theorem (B6) reduces to

$$2\langle K\rangle + \langle W_{ii}\rangle = \oint P_{ik}x_i dS_k. \tag{B17}$$

If the system is at statistical equilibrium, then $\langle K\rangle = \frac{d}{2}Nk_BT$ and $P_{ij}=p\delta_{ij}$ with $p=\Sigma_s\rho_s k_BT/m_s$ [11]. Then, introducing the notation (26) of paper I, we get

$$dNk_BT + \langle W_{ii}\rangle = dPV. \tag{B18}$$

For an ideal gas without self-gravity ($W_{ii}=0$), we recover the perfect gas law $PV=Nk_BT$. Alternatively, for a self-gravitating gas in two dimensions, using Eq. (B15), we get the exact equation of state

$$PV = Nk_B(T - T_c), \tag{B19}$$

with the *exact* critical temperature

$$k_BT_c = \frac{G\sum_{\alpha\neq\beta} m_\alpha m_\beta}{4N}. \tag{B20}$$

For equal-mass particles, we get

$$k_BT_c = \frac{Gm^2}{4}(N-1), \tag{B21}$$

as in Eq. (77). In the mean-field limit

$$k_BT_c = \frac{GM^2}{4N}. \tag{B22}$$

We now consider the strong-friction limit $\xi\to+\infty$. To leading order in $1/\xi$, the N-body distribution is given by

$$P_N(\mathbf{r}_1,\mathbf{v}_1,\ldots,\mathbf{r}_N,\mathbf{v}_N,t) = e^{-\beta\sum_{\alpha=1}^N m_\alpha v_\alpha^2/2}\Phi_N(\mathbf{r}_1,\ldots,\mathbf{r}_N,t) + O(1/\xi), \tag{B23}$$

as can be deduced from the multispecies N-body Fokker-Planck equation [11] generalizing Eq. (4) (this distribution is obtained by requiring that the Fokker-Planck collision term remains finite when $D,\xi\to+\infty$). From this expression, we find that $P_{ij}=p\delta_{ij}$ with $p=\Sigma_s\rho_s k_BT/m_s$ and $\langle K_{ij}\rangle = \frac{1}{2}Nk_BT\delta_{ij}$ even for the out-of-equilibrium problem. From Eq. (B5), we obtain the overdamped virial theorem for a self-gravitating Brownian gas:

$$\frac{1}{2}\xi\langle\dot{I}_{ij}\rangle = \langle W_{ij}\rangle + Nk_BT\delta_{ij} - \frac{1}{2}\oint p(\delta_{ik}x_j + \delta_{jk}x_i)dS_k. \quad \text{(B24)}$$

We can obtain this result from a different manner. In the strong friction limit $\xi \to +\infty$, the inertial term in Eq. (B1) can be neglected so that the stochastic equations of motion reduce to

$$\dot{x}_i^\alpha = \mu_\alpha m_\alpha \sum_{\beta\neq\alpha} \frac{Gm_\beta(x_i^\beta - x_i^\alpha)}{|\mathbf{r}_\beta - \mathbf{r}_\alpha|^d} + \sqrt{2D'_\alpha}R_i^\alpha(t), \quad \text{(B25)}$$

where $D'_\alpha = k_BT\mu_\alpha$ and $\mu_\alpha = 1/\xi m_\alpha$. Taking the derivative of the tensor of inertia (B2) and using Eq. (B25), we get

$$\dot{I}_{ij} = \frac{2}{\xi}W_{ij} + \sum_\alpha m_\alpha\sqrt{2D'_\alpha}[x_i^\alpha R_j^\alpha(t) + x_j^\alpha R_i^\alpha(t)]. \quad \text{(B26)}$$

Now, averaging over the noise using $\langle x_i^\alpha R_j^\alpha\rangle = \sqrt{2D'_\alpha}\delta_{ij}$, and on statistical realizations, we recover Eq. (B24). The scalar virial theorem reads

$$\frac{1}{2}\xi\langle\dot{I}\rangle = dNk_BT + \langle W_{ii}\rangle - dPV, \quad \text{(B27)}$$

where P is defined as in paper I. This is similar to Eq. (32) of paper I obtained from the SP system but now W_{ii} is given by Eq. (B8). In particular, in $d=2$, using Eq. (B13) we obtain

$$\frac{1}{2}\xi\langle\dot{I}\rangle = 2Nk_B(T - T_c) - 2PV, \quad \text{(B28)}$$

with the *exact* critical temperature (B20).

APPENDIX C: DYNAMICAL STABILITY OF HOMOGENEOUS SYSTEMS

In this appendix, we study the linear dynamical stability of a stationary solution of the damped barotropic Euler equations (28) and (29), which is infinite and homogeneous—i.e., $\rho_0(\mathbf{r}) = \rho$ and $\mathbf{u}_0 = \mathbf{0}$. For sake of generality, we consider a potential of the form

$$\Phi(\mathbf{r},t) = \int u(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}',t)d\mathbf{r}', \quad \text{(C1)}$$

where $u(|\mathbf{r}-\mathbf{r}'|)$ is an arbitrary binary potential of the interaction. We shall thus obtain a generalization of the Jeans instability criterion. We note that an infinite homogeneous medium is a stationary solution of the damped barotropic Euler equations provided that it satisfies the condition of hydrostatic balance $\nabla p_0 + \rho_0\nabla\Phi_0 = \mathbf{0}$ which reduces to $\nabla\Phi_0 = \mathbf{0}$. With Eq. (C1) this can be written

$$\int \frac{\partial u}{\partial \mathbf{x}}d\mathbf{x} = \mathbf{0} \quad \text{or} \quad \int u(x)d\mathbf{x} < \infty. \quad \text{(C2)}$$

We shall assume that this condition is fulfilled. We note that for the gravitational potential, this condition is not fulfilled since $\nabla\cdot\int\nabla u d\mathbf{x} = \int \Delta u d\mathbf{x} = S_d G\int\delta(\mathbf{x})d\mathbf{x} = S_d G \neq 0$. Still, the equations for the perturbation are well posed mathematically and, neglecting the above-mentioned inconsistency at zeroth order, can be considered as a first step to investigate the dynamical stability of a gravitational system. This is the so-called *Jeans swindle* [1]. The linearized damped barotropic Euler equations can be written

$$\frac{\partial\delta\rho}{\partial t} + \rho\nabla\cdot\delta\mathbf{u} = 0, \quad \text{(C3)}$$

$$\rho\frac{\partial\delta\mathbf{u}}{\partial t} = -c_s^2\Delta\delta\rho - \rho\Delta\delta\Phi - \xi\rho\delta\mathbf{u}, \quad \text{(C4)}$$

$$\delta\Phi(\mathbf{r},t) = \int u(\mathbf{r}-\mathbf{r}')\delta\rho(\mathbf{r}',t)d\mathbf{r}', \quad \text{(C5)}$$

where we have introduced the velocity of sound $c_s^2 = p'(\rho)$. They can be combined to give

$$\frac{\partial^2\delta\rho}{\partial t^2} + \xi\frac{\partial\delta\rho}{\partial t} = c_s^2\Delta\delta\rho + \rho\Delta\delta\Phi, \quad \text{(C6)}$$

with $\delta\Phi = u * \delta\rho$. Looking for solutions of the form

$$\delta\rho(\mathbf{r},t) = \int \delta\hat{\rho}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}d\mathbf{k}d\omega, \quad \text{(C7)}$$

we obtain the general dispersion relation

$$\omega(\omega + i\xi) = c_s^2k^2 + (2\pi)^d\hat{u}(k)\rho k^2. \quad \text{(C8)}$$

Introducing $\lambda = -i\omega$ and $\hat{v}(k) = -(2\pi)^d\hat{u}(k)$, this can be rewritten

$$\lambda^2 + \xi\lambda - k^2[\rho\hat{v}(k) - c_s^2] = 0. \quad \text{(C9)}$$

The solutions are $\lambda_\pm = \frac{1}{2}(-\xi \pm \sqrt{\Delta})$ with $\Delta(k) = \xi^2 + 4k^2[\rho\hat{v}(k) - c_s^2]$. If $c_s^2 < \rho\hat{v}(k)$, then $\Delta(k) > \xi > 0$ and the system is unstable as $\lambda_+ = \frac{1}{2}(-\xi + \sqrt{\Delta}) > 0$. If $c_s^2 < \rho\hat{v}(k)$, either $\Delta(k) < 0$ and $R_e(\lambda) = -\xi/2$ or $0 < \Delta(k) < \xi$, implying $\lambda_\pm < 0$, so the system is stable. Therefore, the system is stable if

$$c_s^2 > \rho\hat{v}(k) \quad \text{(C10)}$$

and unstable otherwise. For attractive potentials $\hat{v}(k) > 0$, this gives rise to the existence of a critical point as discussed in [12]. Indeed, a necessary condition of instability is that

$$c_s^2 < (c_s^2)_{crit} = \rho\hat{v}(k)_{max}. \quad \text{(C11)}$$

If this condition is fulfilled, the range of unstable wavelengths is determined by

$$\hat{v}(k) > c_s^2/\rho \quad \text{(C12)}$$

and their growth rate is $\lambda_+(k)$. For the Euler equations ($\xi = 0$), the dispersion relation reduces to

$$\omega^2 = c_s^2k^2 + (2\pi)^d\hat{u}(k)\rho k^2. \quad \text{(C13)}$$

In the unstable case the perturbation grows exponentially, and in the stable case the perturbation presents undamped oscillations. In the overdamped limit of the model $\xi \to +\infty$ (Smoluchowski), the dispersion relation reduces to

$$i\xi\omega = c_s^2k^2 + (2\pi)^d\hat{u}(k)\rho k^2, \quad \text{(C14)}$$

which can be obtained directly from the generalized Smoluchowski equation (35). In the unstable case the perturbation

grows exponentially and in the stable case the perturbation decreases exponentially. The intermediate case of finite friction can be treated as in Sec. IV but explicit results demand to specify the potential of interaction.

For the attractive Yukawa potential $\hat{v}(k)=S_dG/(k^2+k_0^2)$ [12], we find that the system is unstable if $c_s^2<(c_s^2)_{crit}=S_dG\rho/k_0^2$ for the wave vectors such that

$$k<k_{max}\equiv\sqrt{\frac{S_dG\rho}{c_s^2}-k_0^2}. \tag{C15}$$

The growth rate $\lambda_+(k)$ is maximum for $k_*^2=(S_dG\rho k_0^2/c_s^2)^{1/2}-k_0^2$, and its value $\lambda_*=\lambda_+(k_*)$ is given by

$$2\lambda_*=-\xi+\sqrt{\xi^2+4S_dG\rho(1-\sqrt{c_s^2k_0^2/S_dG\rho})^2}. \tag{C16}$$

For $c_s=0$ (cold systems), we have $k_{max}=k_*=+\infty$ and $\lambda_*=\frac{1}{2}(-\xi+\sqrt{\xi^2+4S_dG\rho})$. The system is most unstable at small wavelengths. For $k_0=0$ (gravitational potential), we have $(c_s^2)_{crit}=+\infty$, $k_{max}=k_J=(S_dG\rho/c_s^2)^{1/2}$ (Jeans length), $k_*=0$, and $\lambda_*=\frac{1}{2}(-\xi+\sqrt{\xi^2+4S_dG\rho})$. The system is most unstable at large wavelengths. For $\xi=0$ (Euler), $\lambda_*=(S_dG\rho)^{1/2}(1-\sqrt{c_s^2k_0^2/S_dG\rho})$, and for $\xi\to+\infty$ (Smoluchowski), $\lambda_*=(S_dG\rho/\xi)(1-\sqrt{c_s^2k_0^2/S_dG\rho})^2$. Other examples of potentials of interaction are given in [12].

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